**Ch 9 - Introduction to Quantification**

**basic sentences:** noun phrases + verb phrases

**simplest noun phrases:** names, *Max, Claire*

**more complex noun phrases:** common nouns + **determiners**

**determiners:** *every, some, most, the, three, no*

* *every cube*
* *some man from Indiana*
* *most children in the class*
* these are **quantified expressions**
* sentences containing quantified expressions are **quantified sentences**
* such sentences allow us to talk about quantities of things
* quantified sentences have logical properties that depend on the determiner that is used
* validity of arguments depends critically on the quantifiers, that express determiners in English
* to logically analyze arguments that include quantified sentences, we need to be able to model the determiner

**quantification takes us out of the realm of truth-functional connectives**

* **we can’t determine the truth of quantified sentences by looking at the truth values of the constituent sentences**
* *No rich actor is a good actor*
* *Every rich actor is a good actor*
* **these sentences aren’t made up of simpler sentences in the sense that we’ve seen so far**
* **their truth values are determined by the relationship between the collection of rich actors and the collection of good actors**
* propositional logic ignores objects in favor of basic propositions; it doesn’t capture differences in sentences due to different quantifiers

**whenever**

* is an implicit form of quantification meaning *at every time that*
* *Max is home whenever Claire is at the library.*
* ***means*** *Every time when Claire is at the library is a time when Max is home*

**logically implies**

* this is another non-truth-functional connective
* you can’t tell whether P logically implies Q just by looking at the truth table values of P and Q
* a claim with this connective means that every logically possible circumstance that makes P true makes Q true; the claim quantifies over possible circumstances

there are many forms of quantification in English, **but only two forms of quantification are built into FOL explicitly**

* **two quantifier symbols:** ∃, ∀ meaning “something” and “everything” respectively
* using these symbols, plus predicates and truth-functional connectives we can construct phrases like *every cube, three blind mice, no tall student, whenever*
* some quantified expressions are outside the scope of FOL
  + *most students*
  + *many cubes*
  + *infinitely many prime numbers*

**FO logic:** propositional logic extended with new quantifier symbols

**variables**

* variables are a kind of auxiliary symbol
* unlike constants, their semantic function is not to refer to objects, rather they are placeholders that indicate relationships between quantifiers and the argument positions of various predicates, **ie variables only have meaning when used in conjunction with quantifier symbols**
* up to this point individual constants (names) were the only basic terms
* now we have two types of basic terms, variables and individual constants, and we can form complex terms by applying function symbols to either type of basic term
  + *father(max)*
  + *father(x)*
  + *(0+1) x 1*
  + *(y+z) x z*
* with the new terms we can produce expressions that look like atomic sentences, but contain variables in place of names
  + these are called **atomic well-formed formulas** or **atomic wffs**
  + **they are not sentences, but are used in conjunction with quantifier symbols to build sentences**
* **the term sentence is reserved for well-formed formulas in which any variables that do occur are used together with quantifiers that bind them**

**quantifier symbols**

* **always used in conjunction with variables**
* **are called variable binding operators**

**universal quantifier** ∀

* ∀ is used to express **universal claims,** which are expressed in English using quantified phrases such as *everything, each thing, all things,* and *anything*
* always used in connection with a variable, and so is said to be a **variable binding operator**
* **∀x is read “for every object x”, or somewhat misleadingly “for all x”**
* allows us to say that all objects satisfy some condition
* **example**
  + *Everything is at home.*
  + ∀x Home(x)
  + every object x meets the condition “x is at home”
  + *Every doctor is smart.*
  + ∀x (Doctor(x) -> Smart(x))

**existential quantifier** ∃

* symbol used to express existential claims, expressed in English using *something, at least one thing, a, an*
* always used in connection with a variable, so is also a variable binding operator
* **∃x is read “for some object x or (somewhat misleadingly) “for some x”**
* allows us to say that at least one object satisfies some condition
* **examples**
  + *Something is at home.*
  + ∃x Home(x)
  + *Some doctor is smart.*
  + ∃x (Doctor(x) & Smart(x))

**wffs and sentences**

* **atomic wff:** any n-ary predicate followed by n terms, where terms can now contains either variables or individual constants
* we build more complicated wffs by applying the following rules to atomic wffs

1. if P is a wff, so is ~P
2. if P1,...,Pn are wffs, so is P1 & … & Pn
3. if P1,...,Pn are wffs, so is P1 | … | Pn
4. if P and Q are wffs, so it P → Q
5. if P and Q are wffs, so is P ↔ Q
6. if P is a wff and v a variable, then ∀v P is a wff
7. if P is a wff and v a variable, then ∃v P is a wff

**example**

atomic wffs: Cube(x), Small(x)

apply rule 2: Cube(x) & Small(x)

we now have another wff

**free and bound variables**

1. Any variable in an atomic wff is free or unbound
2. the free variables in P are also free in ~P
3. the free variables in P1,...,Pn are all free in P1 & … & Pn
4. the free variables in P1,...,Pn are all free in P1 | … | Pn
5. the free variables in P and Q are all free in P → Q
6. the free variables in P and Q are all free in P ↔ Q
7. all of the free variables in P are free in ∀v P, except for v, and every occurrence of v in P is said to be bound
8. all of the free variables in P are free in ∃v P, except for v, and every occurrence of v in P is said to be bound

**example**

(Cube(x) & Small(x)) → ∃y LeftOf(x,y)

* variable y is bound by the quantifier ∃y
* variable x is free

**sentence:** a wff with no free variables

∀x (Cube(x) & Small(x)) → ∃y LeftOf(x,y)

is a sentence: has no free variables

**tricky example**

∃x Doctor(x) & Smart(x)

* the last occurrence of x is free
* the existential quantifier is applied to the predicate Doctor(x), binding only this occurrence of x
* parentheses tell you what the scope of a quantifier is: which variables fall under its influence and which don’t
* **it is really an occurrence of a variable that is either free or bound, not the variable itself**

**Semantics for the quantifiers**

* we don’t have rules yet to determine the truth value of quantified sentences
* the expression to which we apply the quantifier in order to build a sentence is usually not itself a sentence
* ∃x Cube(x) has a quantifier binding the variable x, a unary predicate with a variable as an argument
* Cube(x) on its own is not a sentence: it contains a free variable. It is neither true or false

**satisfaction**

* an object satisfies the atomic wff Cube(x) if and only if the object is a cube
* an object satisfies the complex wff Cube(x) & Small(x) if and only if it is both a cube and small

**how Tarski’s World checks truth of quantified sentences**

* S(x) a wff containing x as its only free variable
* we want to know if a given object, with name b, satisfied the wff
* form a new sentence S(b) by replacing all free occurrences of x by the individual constant b
* if S(b) is true, then the object satisfies the formula S(x); if not then it doesn’t satisfy the formula

**given this notion of satisfaction, we can describe when the sentence ∃x Cube(x) is true, and when the sentence ∀x Cube(x) is true**

* in the first case, it is true if and only if there is at least one object that satisfied S(x)
* in the second case, it is true if and only if every object satisfied S(x)

**A sentence of the form ∃x S(x) is true if and only if at least one object satisfies S(x)**

**A sentence of the form ∀x S(x) is true if and only if every object satisfies S(x)**

**Therefore, the truth of a quantified sentence is determined in terms of the objects that satisfy the embedded wff**

**What objects are considered?** So far we have assumed there is a clear, non-empty collection of objects we are talking about

* in general sentences containing quantifiers are only true or false relative to some **domain of discourse** or **domain of quantification**
* sometimes the intended domain contains all objects there are
* sometimes the intended domain is much more restricted collection of things, say the people in the room, some particular set of physical objects, or some collection of numbers
* we will specify the domain explicitly
* in FOl we always assume the domain of discourse contains at least one object and that every individual constant in the language stands for an object in that domain

**notation**

* we have used P and Q to stand for possibly complex sentence of propositional logic
* we will use S(x) or P(y) to stand for a possibly complex wff of FOL
* when we write P(b) this stands for the result of replacing all free occurrences of y by the individual constant b

**example**

P(y) is ∃x (LeftOf(x,y) & RightOf(x,y))

P(b) is ∃x (LeftOf(x,b) & RightOf(x,b))

In P(b), the b replaces only free occurrences of the variable.

In P(y) the variable in parentheses represents the free variables in the wff

**The four Aristotelian forms**

* Aristotle studied the kinds of reasoning associated with quantified noun phrases such as *every man, no man, some man.* These are expressions we would translate with quantifier symbols
* Four main sentence forms in Aristotle’s logic
  + *All P’s are Q’s.* **∀x P(x)** → **Q(x)**
  + *Some P’s are Q’s.* **∃x P(x) & Q(x)**
  + *No P’s are Q’s.* **~(∃x P(x) & Q(x))**
    - **alternatively, ∀x P(x)** → **~Q(x)**
  + *Some P’s are not Q’s.* **∃x (P(x) & ~Q(x))**
* these are the very simplest sorts of sentences built using quantifiers
* we should remember them well, as more complicated forms are elaborations of these

**translating complex noun phrases**

* complex noun phrases
  + a boy living in Omaha
  + every girl living in Duluth

**existential noun phrases**

* The first of these has a translation that usually involves the existential quantifier
  + typically such noun phrases start with one of the determiners *some, a,* and *an,* including *something*
  + these are called **existential noun phrases** since they typically assert the existence of something
* ***A small, happy dog is at home.***
  + **∃x ((Small(x) & Happy(x) & Dog(x)) & Home(x))**
* general pattern: ∃ frequently together with &

**universal noun phrases**

* begin with determiners like *every, each,* and *all*
* usually translated with universal quantifier
* sometimes can begin with *no* and *any*
* general pattern: ∀ frequently together with →
* ***Every small dog that is at home is happy.***
  + ∀x ((Dog(x) & Small(x) & Home(x)) → Happy(x))

**tricky and problematic cases**

* **vacuously true generalization**
  + ∀x (P(x) → Q(x))
  + in a world where there are no objects satisfying P(x)
  + the sentence is true because no objects satisfy the antecedent
* **inherently vacuous generalizations**
  + sentences that can only be true if they are vacuously true
  + ∀x (Tet(x) → Cube(x))
  + If there were an object satisfying the antecedent, the whole sentence would be a contradiction (false in every logically possible situation)
* if a professor says **Every freshman who took my class got an A.** but no freshman took her class, then the sentence is vacuously true.
* There is an implicature at work “there were freshman in the class”.
* Given the implicature, the statement seems misleading.
* Note that the statement could be non-vacuously true; ie it could be true without being perceived as misleading.
* In the case of an inherently vacuous claim, they are only true when they are misleading.

**term:** variables and individual constants are terms of an FOL; also, the results of combining an n-ary function symbol f with n terms forms a new term

**determiners:** *every, some, most, the, three, no*

**quantified expressions:** expressions with determiners

**quantified sentences:** sentences containing quantified expressions

**variables:** placeholders that indicate relationships between quantifiers and the argument positions of various predicates

**well-formed formulas:** expressions (predicates, function symbols) that contain names or variables

* wffs are a class of expressions
* are built from terms and predicate symbols

**atomic wff:** n-ary predicate followed by n terms where the terms contain either variables or individual constants

**complex wff:** formed from atomic or complex wffs by applying certain rules to such wffs; e.g. from n wffs, we can form a complex wff that is the conjunction of the n initial wffs

**sentence:** wff in which all variables are bound (by quantifiers)

**quantifier:** formal form of quantification in FOL; there are only two in FOL: ∃, ∀